

ON SOME NEW MODEL CATEGORY STRUCTURES FROM OLD, ON THE SAME UNDERLYING CATEGORY

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ABSTRACT. We make a study of $\ell\ell$ -extensions of model category structures. We prove an existence result of $\ell\ell$ -extensions, present some specific and some rather formal results about them and give an application of the existence result to the homotopy theory of categories enriched over a monoidal model category.

Given a category and a model structure on it, we inadequately say that an *extension* of the model structure is a model structure on the same category having more weak equivalences, and that an $\ell\ell$ -*extension* (or, *extension of type $\ell\ell$*) of the model structure is an extension which has *less* cofibrations and *less* fibrations than the given one. We allow an $\ell\ell$ -extension to have the same cofibrations or fibrations as the given model structure.

Every category having suitable limits and colimits admits a minimal model structure in which the weak equivalences are the isomorphisms and all maps are cofibrations as well as fibrations. Any other model structure on it is an $\ell\ell$ -extension of the minimal model structure. $\ell\ell$ -extensions arise in disparate places such as the theory of E^2 model categories of Dwyer-Kan-Stover [3] or the homotopy theory of (multi)categories enriched over monoidal model categories or of precategories enriched over cartesian model categories, for example.

In this paper we make a study of $\ell\ell$ -extensions of model structures. Our interest in them comes, in part, from the homotopy theory of categories enriched over monoidal model categories. Our approach to the study of $\ell\ell$ -extensions is mainly influenced by Bousfield's work [2]. His work is a vast generalization of [3] and of others (see the Introduction to [2]). The main result of the present work, Theorem 1.2, can be seen as a generalization of [2, Theorem 3.3] tailored to capture a common feature of all of these homotopy theories.

The paper is organized as follows. In Section 1 we offer an existence result of $\ell\ell$ -extensions (see Theorem 1.2). In Section 2 we present some results about $\ell\ell$ -extensions, some of them which are in a specific context. Sections 1 and 2 are independent of each other. In Section 3 we give an application of our existence result of $\ell\ell$ -extensions to the homotopy theory of categories enriched over a monoidal model category (see Theorem 3.1).

1. AN EXISTENCE RESULT OF $\ell\ell$ -EXTENSIONS

Let (W, C, F) be a model structure on a category \mathbf{M} . W stands for the class of weak equivalences, C for the class of cofibrations and F for the class of fibrations. Let $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ be three classes of maps of \mathbf{M} such that $W \subset W^{\mathcal{G}}, C^{\mathcal{G}} \subseteq C$

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and $F^{\mathcal{G}} \subseteq F$. In this section we give sufficient conditions for $(W^{\mathcal{G}}, C^{\mathcal{G}}, F^{\mathcal{G}})$ to form a model structure on \mathbf{M} with $W^{\mathcal{G}}$ as the class of weak equivalences, $C^{\mathcal{G}}$ as the class of cofibrations and $F^{\mathcal{G}}$ as the class of fibrations. The result is stated as Theorem 1.2.

Our approach is heavily influenced by the proof of [2, Theorem 3.3]. However, there are differences. For example, it will follow from our result that everything after [2, Proposition 3.17] which pertains to the proof of [2, Theorem 3.3] is formal. This is somehow implicit in [2]. We make it explicit in a way that uses less assumptions; this difference will turn out to be essential for the application that we have in mind (see Section 3). Our approach also encompasses the proof of [2, Theorem 12.4].

The next result sets the stage.

Lemma 1.1. *Let W , C and F be three classes of maps of a category \mathbf{M} with pushouts. We make the following assumptions.*

- (1) *W has the two out of three property.*
- (2) *C is closed under compositions and pushouts.*
- (3) *For every commutative solid arrow diagram in \mathbf{M}*

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where j is in $C \cap W$ and p is in F , there is a dotted arrow making everything commute.

- (4) *Every map f of \mathbf{M} factors as $f = qi$, where i is a map in C and q is a map in $F \cap W$.*

Then for every commutative solid arrow diagram in \mathbf{M}

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow q \\ B & \longrightarrow & Y \end{array}$$

where i is in C and q is in $F \cap W$, there is a dotted arrow making everything commute.

Proof. For the first part, we construct a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & D & \longrightarrow & X \\ i \downarrow & & \downarrow j & & \downarrow q \\ B & \longrightarrow & F & \longrightarrow & Y \end{array}$$

with j in $C \cap W$ and then apply (3) to the right square diagram. Factor (4) the map $A \rightarrow X$ into a map $A \rightarrow D$ in C followed by a map $D \rightarrow X$ in $F \cap W$. Let E be the pushout of $A \rightarrow D$ along i . By (2) the map $D \rightarrow E$ is in C . Factor (4) the canonical map $E \rightarrow Y$ into a map $E \rightarrow F$ in C followed by a map $F \rightarrow Y$ in $F \cap W$. Let j be the composite $D \rightarrow E \rightarrow F$. j is in C by (2) and in W by (1). \square

The initial object of a category, when it exists, is denoted by \emptyset .

Theorem 1.2. (after Bousfield) *Let (W, C, F) be a model structure on a finitely bicomplete category \mathbf{M} . Let $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ be three classes of maps of \mathbf{M} such that $W \subset W^{\mathcal{G}}, C^{\mathcal{G}} \subseteq C$ and $F^{\mathcal{G}} \subseteq F$. We make the following assumptions.*

- (1) $W^{\mathcal{G}}$ has the two out of three property.
- (2) $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ are closed under retracts.
- (3) $C^{\mathcal{G}}$ is closed under compositions and pushouts.
- (4) For every object X of \mathbf{M} , the map $\emptyset \rightarrow X$ is in $C^{\mathcal{G}}$ if and only if X is cofibrant.
- (5) Every cofibrant object of \mathbf{M} has an $\mathbf{M}^{\mathcal{G}}$ -cylinder object. This means that for every cofibrant object X of \mathbf{M} there is a factorization

$$X \sqcup X \xrightarrow{i_0 \sqcup i_1} \text{Cyl}X \xrightarrow{p} X$$

of the folding map $X \sqcup X \rightarrow X$ such that $i_0 \sqcup i_1$ is in $C^{\mathcal{G}}$ and p is in $W^{\mathcal{G}}$.

- (6) W is closed under pushouts along maps from $C^{\mathcal{G}}$ between cofibrant objects.
- (7) For every commutative solid arrow diagram in \mathbf{M}

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where j is in $C^{\mathcal{G}} \cap W^{\mathcal{G}}$ and p is in $F^{\mathcal{G}}$, there is a dotted arrow making everything commute.

- (8) Every map f of \mathbf{M} factors as $f = pj$, where j is a map in $C^{\mathcal{G}} \cap W^{\mathcal{G}}$ and p is a map in $F^{\mathcal{G}}$.

Then the classes $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ form a model structure on \mathbf{M} .

Proof. Suppose we have shown that every map f of \mathbf{M} factors as $f = qi$, where i is a map in $C^{\mathcal{G}}$ and q is a map in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$. Then, since we have (7), (8) and (3), the fact that $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ form a model structure would follow from Lemma 1.1.

To establish the desired factorization, let first $g : X \rightarrow Y$ be a map between cofibrant objects. We shall construct the mapping cylinder factorization $g = p_g i_g$ of g , where i_g is in $C^{\mathcal{G}}$ and p_g in $W^{\mathcal{G}}$. Let $X \sqcup X \xrightarrow{i_0 \sqcup i_1} \text{Cyl}X \xrightarrow{p} X$ be an $\mathbf{M}^{\mathcal{G}}$ -cylinder object for X (5). Consider the following diagram, in which all squares are pushouts

$$\begin{array}{ccccccc} \emptyset & \longrightarrow & X & & & & \\ \downarrow & & \downarrow i_1 & & & & \\ X & \xrightarrow{i_0} & X \sqcup X & \xrightarrow{i_0 \sqcup i_1} & \text{Cyl}X & \xrightarrow{p} & X \\ \downarrow g & & \downarrow X \sqcup g & & \downarrow \pi_g & & \\ Y & \xrightarrow{\sigma_Y} & X \sqcup Y & \xrightarrow{h} & M_f & & \end{array}$$

Put $i_g = h(X \sqcup g)i_1$ and $j_g = h\sigma_Y$. There is a unique map $p_g : M_f \rightarrow Y$ such that $gp = p_g\pi_g$ and $p_g j_g = 1_Y$. Then $g = p_g i_g$. The map i_g is in $C^{\mathcal{G}}$ by (3) and (4). The map j_g is in $C^{\mathcal{G}} \cap W^{\mathcal{G}}$ using (1), (2), (7) and (8). Therefore p_g is in $W^{\mathcal{G}}$ by (1).

Let now $f : X \rightarrow Y$ be an arbitrary map of \mathbf{M} . We can construct a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{u} & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \xrightarrow{v} & Y \end{array}$$

in which u and v are in W and \tilde{X} and \tilde{Y} are cofibrant. By the above, \tilde{f} factors as $\tilde{f} = p_{\tilde{f}} i_{\tilde{f}}$. Let D be the pushout of $i_{\tilde{f}} : \tilde{X} \rightarrow M_{\tilde{f}}$ along u . By (3) the map $X \rightarrow D$ is in $C^{\mathcal{G}}$. By (6) the map $M_{\tilde{f}} \rightarrow D$ is in W . Factor (8) the canonical map $D \rightarrow Y$ into a map $D \rightarrow E$ in $C^{\mathcal{G}} \cap W^{\mathcal{G}}$ followed by map $q : E \rightarrow Y$ in $F^{\mathcal{G}}$. q is in $W^{\mathcal{G}}$ by (1). Take i to be the composite $X \rightarrow D \rightarrow E$, then the desired factorization is $f = qi$. \square

We denote by $\mathbf{M}^{\mathcal{G}}$ the model structure constructed in Theorem 1.2. By construction, the cofibrant objects of $\mathbf{M}^{\mathcal{G}}$ coincide with the cofibrant objects of \mathbf{M} .

Proposition 1.3. *The model category $\mathbf{M}^{\mathcal{G}}$ is left proper.*

Proof. The proof proceeds exactly as in [2, Proposition 3.27]. For completeness we repeat it. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

be a pushout diagram in \mathbf{M} in which i is in $C^{\mathcal{G}}$ and f in $W^{\mathcal{G}}$. As in *loc. cit.* we may assume that A and B are cofibrant. Factor f as a map $A \rightarrow \tilde{X}$ in C followed by a map $\tilde{X} \rightarrow Y$ in $F \cap W$ and then take consecutive pushouts. The first map that factors $B \rightarrow Y$ is in $W^{\mathcal{G}}$ as in *loc. cit.*, the second map is in $W^{\mathcal{G}}$ by assumption (6). \square

To make the connection between Theorem 1.2 and [2, Theorems 3.3 or 12.4], let \mathcal{C} be a model category as in *loc. cit.*. We take $\mathbf{M} = c\mathcal{C}$ with the Reedy model structure, $W^{\mathcal{G}}$ to be the class of \mathcal{G} -equivalences, $C^{\mathcal{G}}$ the class of \mathcal{G} -cofibrations and $F^{\mathcal{G}}$ the class of \mathcal{G} -fibrations.

Theorem 1.2 certainly admits variations. We note one of them.

Proposition 1.4. *Let (W, C, F) be a model structure on a finitely bicomplete category \mathbf{M} . Let W' and $W^{\mathcal{G}}$ be two classes of maps of \mathbf{M} such that $W \subset W' \subseteq W^{\mathcal{G}}$. We define $F^{\mathcal{G}}$ to be the class of maps of \mathbf{M} having the right lifting property with respect to the maps in $C \cap W'$, and $C^{\mathcal{G}}$ to be the class of maps of \mathbf{M} having the left lifting property with respect to the maps in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$. We make the following assumptions.*

- (1) W' and $W^{\mathcal{G}}$ have the two out of three property.
- (2) W' and $W^{\mathcal{G}}$ are closed under retracts.
- (3) Every map f of \mathbf{M} factors as $f = pj$, where j is a map in $C \cap W'$ and p is a map in $F^{\mathcal{G}}$.
- (4) Every map f of \mathbf{M} factors as $f = qi$, where i is a map in $C^{\mathcal{G}}$ and q is a map in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$.

Then the classes $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ form a model structure on \mathbf{M} .

Theorem 1.2 admits a dual formulation. For future reference we state it below. The terminal object of a category, when it exists, is denoted by $*$.

Theorem 1.5. *Let (W, C, F) be a model structure on a finitely bicomplete category \mathbf{M} . Let $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ be three classes of maps of \mathbf{M} such that $W \subset W^{\mathcal{G}}, C^{\mathcal{G}} \subseteq C$ and $F^{\mathcal{G}} \subseteq F$. We make the following assumptions.*

- (1) $W^{\mathcal{G}}$ has the two out of three property.
- (2) $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ are closed under retracts.
- (3) $F^{\mathcal{G}}$ is closed under compositions and pullbacks.
- (4) For every object X of \mathbf{M} , the map $X \rightarrow *$ is in $F^{\mathcal{G}}$ if and only if X is fibrant.
- (5) Every fibrant object of \mathbf{M} has an $\mathbf{M}^{\mathcal{G}}$ -path object. This means that for every fibrant object X of \mathbf{M} there is a factorization

$$X \xrightarrow{s} \text{Path}X \xrightarrow{p_0 \times p_1} X \times X$$

of the diagonal map $X \rightarrow X \times X$ such that s is in $W^{\mathcal{G}}$ and $p_0 \times p_1$ is in $F^{\mathcal{G}}$.

- (6) W is closed under pullbacks along maps from $F^{\mathcal{G}}$ between fibrant objects.
- (7) For every commutative solid arrow diagram in \mathbf{M}

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & \nearrow & \downarrow q \\ B & \longrightarrow & Y \end{array}$$

where i is in $C^{\mathcal{G}}$ and q is in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$, there is a dotted arrow making everything commute.

- (8) Every map f of \mathbf{M} factors as $f = qi$, where i is a map in $C^{\mathcal{G}}$ and q is a map in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$.

Then the classes $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ form a right proper model structure on \mathbf{M} .

Motivated by the previous considerations and other naturally occurring examples we make the following

Definition 1. Let (W, C, F) be a model structure on a category \mathbf{M} . An $\ell\ell$ -**extension** (or, **extension of type $\ell\ell$**) of (W, C, F) is a model structure $(W^{\mathcal{G}}, C^{\mathcal{G}}, F^{\mathcal{G}})$ on \mathbf{M} such that $W \subset W^{\mathcal{G}}, C^{\mathcal{G}} \subseteq C$ and $F^{\mathcal{G}} \subseteq F$.

Every model structure on a category is an $\ell\ell$ -extension of its minimal model structure (W =isomorphisms, C =all maps, F =all maps). Thus, Theorem 1.2 gives, in particular, a way to construct model categories with all objects cofibrant.

An $\ell\ell$ -extension as in Definition 1 for which $C^{\mathcal{G}} = C$ is sometimes called left Bousfield localization, and one for which $F^{\mathcal{G}} = F$ is sometimes called right Bousfield localization. Left and right Bousfield localizations are ubiquitous [5].

There are other kinds of extensions. For example, given a category and a model structure on it, an ℓm -**extension** of the given model structure is another model structure on the same category having more weak equivalences, less cofibrations and more fibrations. The following existence result of ℓm -extensions can be proved in a similar way as (the dual of) Theorem 1.2.

Theorem 1.6. *Let (W, C, F) be a model structure on a finitely bicomplete category \mathbf{M} . Let $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ be three classes of maps of \mathbf{M} such that $W \subset W^{\mathcal{G}}, C^{\mathcal{G}} \subseteq C$ and $F \subset F^{\mathcal{G}}$. We make the following assumptions.*

- (1) $W^{\mathcal{G}}$ has the two out of three property.
- (2) $W^{\mathcal{G}}$, $C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ are closed under retracts.
- (3) $F^{\mathcal{G}}$ is closed under compositions and pullbacks.
- (4) For every commutative solid arrow diagram in \mathbf{M}

$$\begin{array}{ccc} A & \xrightarrow{\quad} & X \\ i \downarrow & \nearrow & \downarrow q \\ B & \xrightarrow{\quad} & Y \end{array}$$

where i is in $C^{\mathcal{G}}$ and q is in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$, there is a dotted arrow making everything commute.

- (5) Every map f of \mathbf{M} factors as $f = qi$, where i is a map in $C^{\mathcal{G}}$ and q is a map in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$.

- (6) The model structure on \mathbf{M} is right proper.
Then the classes $W^{\mathcal{G}}$, $C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ form a model structure on \mathbf{M} .

2. ON RIGHT DERIVED FUNCTORS AND COMPLETIONS, AND OTHER FORMAL RESULTS

There seem to be few results about $\ell\ell$ -extensions. In the first part of this section we make an attempt to view results like Theorem 6.2 and the first part of Theorem 6.5 from [2]—or rather their generalizations, as explained in [2, 6.20 and 12.8], as results about (very specific) $\ell\ell$ -extensions. We hope that our approach highlights both general and particular aspects of this part of Bousfield's work. Inspired by [5, Chapters 3,4,5 and 9], we study in the second part of this section the behaviour of some model category theoretical properties under the passage to an $\ell\ell$ -extension.

Throughout, \mathbf{M} is a bicomplete category and $(W^{\mathcal{G}}, C^{\mathcal{G}}, F^{\mathcal{G}})$ an $\ell\ell$ -extension of a model structure (W, C, F) on \mathbf{M} . We denote by $\mathbf{M}^{\mathcal{G}}$ the $\ell\ell$ -extension. The fibrant (cofibrant) objects of $\mathbf{M}^{\mathcal{G}}$ will be referred to as \mathcal{G} -fibrant (\mathcal{G} -cofibrant). The fibrant (cofibrant) objects with respect to the model structure (W, C, F) will be simply referred to as fibrant (cofibrant).

Proposition 2.1. *Suppose that the \mathcal{G} -cofibrant objects coincide with the cofibrant objects of \mathbf{M} and that $\mathbf{M}^{\mathcal{G}}$ is a simplicial model category. Let \mathcal{N} be a simplicial category and $T : \mathbf{M} \rightarrow \mathcal{N}$ a simplicial functor with the property that T sends the maps in W between fibrant objects to isomorphisms. Let $H^s : \mathcal{N} \rightarrow \mathcal{N}'$ be a functor which identifies strictly simplicially homotopic maps. Then the composite $H^s T$ sends the maps in $W^{\mathcal{G}}$ between \mathcal{G} -fibrant objects to isomorphisms.*

Proof. Let $f : X \rightarrow Y$ be a map in $W^{\mathcal{G}}$ between \mathcal{G} -fibrant objects. We can construct a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{u} & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \xrightarrow{v} & Y \end{array}$$

in which u and v are in $F \cap W$ and \tilde{X} and \tilde{Y} are cofibrant. Since $F \cap W \subseteq F^{\mathcal{G}} \cap W^{\mathcal{G}}$, it follows that \tilde{X} and \tilde{Y} are \mathcal{G} -fibrant. Thus, $T(\tilde{f})$ is a strict simplicial homotopy equivalence using [5, Proposition 9.5.24(2)], and therefore $H^s T(\tilde{f})$ is an isomorphism. \square

Proposition 2.1 implies that the right derived functor $\mathcal{R}_G H^s T$ of $H^s T$ with respect to \mathbf{M}^G exists. We shall describe a (very particular) way to compute it.

Let \mathcal{G}' be a class of objects of \mathbf{M} which is invariant under W . That is, if $X \rightarrow Y$ is in W , then $X \in \mathcal{G}'$ if and only if $Y \in \mathcal{G}'$. We assume that every \mathcal{G} -fibrant object is in \mathcal{G}' . A **weak \mathcal{G} -fibrant approximation** to an object A of \mathbf{M} is a diagram $A \xrightarrow{j} Y$, where j is in W^G and Y is in \mathcal{G}' .

Let $cst : \mathbf{M} \rightarrow \mathbf{M}^\Delta$ be the constant cosimplicial object functor. For an object $Y \in \mathbf{M}$, let \bar{Y} be a Reedy fibrant approximation to $cst Y$ in $(\mathbf{M}^G)^\Delta$, and let $\bar{\bar{Y}} = Tot \bar{Y}$. We have an induced map $\alpha : Y \rightarrow \bar{\bar{Y}}$.

Lemma 2.2. *Suppose that the \mathcal{G} -cofibrant objects coincide with the cofibrant objects of \mathbf{M} and that \mathbf{M}^G is a simplicial model category. Let \mathcal{N} be a simplicial category and $T : \mathbf{M} \rightarrow \mathcal{N}$ a simplicial functor with the property that T sends the maps in W to isomorphisms. Let $H^s : \mathcal{N} \rightarrow \mathcal{N}'$ be a functor which identifies strictly simplicially homotopic maps. Suppose furthermore that for each fibrant object Y of \mathbf{M} which belongs to \mathcal{G}' , the map α is in W^G and the map $H^s T(\alpha)$ is an isomorphism. Then $\mathcal{R}_G H^s T$ can be computed using weak \mathcal{G} -fibrant approximations.*

Proof. Let A be an object of \mathbf{M} . Let $A \rightarrow \bar{A}$ be a map in $C^G \cap W^G$ with \bar{A} \mathcal{G} -fibrant and $A \rightarrow Y$ a weak \mathcal{G} -fibrant approximation to A . Let $Y \rightarrow \underline{Y}$ be a fibrant approximation to Y . The diagram

$$\begin{array}{ccccccc} A & \longrightarrow & Y & \longrightarrow & \underline{Y} & \xrightarrow{\alpha} & \bar{\bar{Y}} \\ \downarrow & & & & & & \\ \bar{A} & & & & & & \end{array}$$

has a lifting $\bar{A} \rightarrow \bar{\bar{Y}}$. By Proposition 2.1 and assumptions it follows that $\mathcal{R}_G H^s T A \cong H^s T Y$. □

To make the connection between Lemma 2.2 and [2, Theorem 6.2], we take $\mathbf{M} = c\mathcal{C}$ and \mathcal{G}' to be the class of termwise \mathcal{G} -injective objects [2, Definition 6.1]. Then a weak \mathcal{G} -fibrant approximation is just a weak \mathcal{G} -resolution as in *loc. cit.*

Proposition 2.3. *Suppose that the \mathcal{G} -cofibrant objects coincide with the cofibrant objects of \mathbf{M} and that \mathbf{M}^G is a simplicial model category. Let \mathcal{N} be a simplicial model category and $T : \mathbf{M} \rightarrow \mathcal{N}$ a simplicial functor with the property that T sends the maps in W between fibrant objects to weak equivalences. Then T sends the maps in W^G between \mathcal{G} -fibrant objects to weak equivalences.*

Proof. The proof is the same as for Proposition 2.1, using now the fact that a simplicial functor between simplicial model categories sends weak equivalences between cofibrant-fibrant objects to weak equivalences. □

Proposition 2.3 implies that the total right derived functor $\mathcal{R}_G T$ of T with respect to \mathbf{M}^G exists. We shall describe a (very particular) way to compute it.

Lemma 2.4. *Suppose that the \mathcal{G} -cofibrant objects coincide with the cofibrant objects of \mathbf{M} and that \mathbf{M}^G is a simplicial model category. Let \mathcal{N} be a simplicial model category and $T : \mathbf{M} \rightarrow \mathcal{N}$ a simplicial functor with the property that T sends the maps in W between fibrant objects to weak equivalences. Suppose furthermore that*

for each fibrant object Y of \mathbf{M} which belongs to \mathcal{G}' , the map α is in $W^{\mathcal{G}}$ and the map $T(\alpha)$ is a weak equivalence. Let $A \rightarrow Y$ be a weak \mathcal{G} -fibrant approximation to an object A . Then $\mathcal{R}_{\mathcal{G}}TA \cong \mathcal{R}TY$, where $\mathcal{R}T$ is the total right derived functor of T .

Proof. The proof is the same as for Lemma 2.2, using Proposition 2.3. \square

To make the connection between Lemma 2.4 and the first part of [2, Theorem 6.5], we take $\mathbf{M} = c\mathcal{C}$, \mathcal{G}' to be the class of termwise \mathcal{G} -injective objects, $\mathcal{N} = \mathcal{C}$ and $T = Tot$.

The next result can be seen as a generalization of [5, Proposition 3.4.4].

Proposition 2.5. (1) If \mathbf{M} is left proper and the cofibrant objects coincide with the \mathcal{G} -cofibrant objects, then $\mathbf{M}^{\mathcal{G}}$ is left proper.

(2) If \mathbf{M} is right proper and the fibrant objects coincide with the \mathcal{G} -fibrant objects, then $\mathbf{M}^{\mathcal{G}}$ is right proper.

Proof. The proof of (1) is similar to the proof of Proposition 1.3. The proof of (2) is dual. \square

The next result can be seen as a generalization of [5, Propositions 3.3.15 and 3.4.6]. To state it, we introduce some terminology. Let \mathbf{M}_0 and \mathbf{M}_1 be two full subcategories of \mathbf{M} with $\mathbf{M}_0 \subseteq \mathbf{M}_1$. We say that \mathbf{M}_0 is **invariant in \mathbf{M}_1 under W** if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{v} & B' \end{array}$$

in which u and v are in W and f and g are in \mathbf{M}_1 , f is in \mathbf{M}_0 if and only if g is in \mathbf{M}_0 .

Proposition 2.6. (1) Let

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & Z & \end{array}$$

be a commutative diagram in \mathbf{M} with $f \in F$, $g \in F^{\mathcal{G}}$ and $h \in W$. Then $f \in F^{\mathcal{G}}$. If \mathbf{M} is left proper and $C^{\mathcal{G}}$ is invariant in C under W , then the converse holds, that is, $f \in F^{\mathcal{G}}$ and $g \in F$ imply $g \in F^{\mathcal{G}}$.

(2) Let

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow g \\ B & \xrightarrow{h} & C \end{array}$$

be a commutative diagram in \mathbf{M} with $f \in C^{\mathcal{G}}$, $g \in C$, and $h \in W$. Then $g \in C^{\mathcal{G}}$. If \mathbf{M} is right proper and $F^{\mathcal{G}}$ is invariant in F under W , then the converse holds, that is, $g \in C^{\mathcal{G}}$ and $f \in C$ imply $f \in C^{\mathcal{G}}$.

Proof. We will prove (1); the proof of (2) is dual. We prove the first part. Factor the map h as a map $X \rightarrow E$ in \mathbf{C} followed by a map $q : E \rightarrow Y$ in $\mathbf{F} \cap \mathbf{W}$. The diagram

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & & \\ \downarrow & & \downarrow f & & \\ E & \xrightarrow{q} & Y & \xrightarrow{g} & Z \end{array}$$

has a lifting, so f is a retract of gq . But $gq \in \mathbf{F}^{\mathcal{G}}$.

We now prove the converse. We will show that every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{t} & Y \\ i \downarrow & & \downarrow g \\ B & \xrightarrow{u} & Z \end{array}$$

where i is in $\mathbf{C}^{\mathcal{G}} \cap \mathbf{W}^{\mathcal{G}}$ has a lifting. We can construct a commutative diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{q} & A \\ \tilde{i} \downarrow & & \downarrow i \\ \tilde{B} & \xrightarrow{r} & B \end{array}$$

in which q and r are in \mathbf{W} , \tilde{A} and \tilde{B} are cofibrant and \tilde{i} is a cofibration. By assumption and [5, Proposition 13.2.1(1)] we may assume without loss of generality that i has cofibrant domain to begin with. Then the proof proceeds exactly as in [5, Proposition 3.4.6(1)]. \square

Let \mathbf{N} be another model category and let $S : \mathbf{M} \rightleftarrows \mathbf{N} : T$ be a Quillen pair in which S is the left adjoint. Let $\mathbf{N}^{\mathcal{G}}$ be an $\ell\ell$ -extension of \mathbf{N} . The fibrant objects of $\mathbf{N}^{\mathcal{G}}$ will be referred to as \mathcal{G} -fibrant. We assume that the adjoint pair (S, T) is also a Quillen pair with respect to the model structures $\mathbf{M}^{\mathcal{G}}$ and $\mathbf{N}^{\mathcal{G}}$. As such, we denote it by $(S^{\mathcal{G}}, T^{\mathcal{G}})$.

Proposition 2.7. (1) *Suppose that the total right derived functor of T is full and faithful. If the \mathcal{G} -cofibrant objects of \mathbf{M} coincide with the cofibrant objects of \mathbf{M} , then the total right derived functor of $T^{\mathcal{G}}$ is full and faithful.*

(2) *Suppose that the total left derived functor of S is full and faithful. If the \mathcal{G} -fibrant objects of \mathbf{N} coincide with the fibrant objects of \mathbf{N} , then the total left derived functor of $S^{\mathcal{G}}$ is full and faithful.*

Proof. We will prove (1); the proof of (2) is dual. It is sufficient to prove that for every \mathcal{G} -fibrant object X of \mathbf{N} and for some cofibrant approximation $\tilde{C}TX$ to TX in $\mathbf{M}^{\mathcal{G}}$, the composite map $S\tilde{C}TX \rightarrow STX \rightarrow X$ is in $\mathbf{W}^{\mathcal{G}}$. Let $\tilde{C}TX$ be any cofibrant approximation to TX in \mathbf{M} . Since a \mathcal{G} -fibrant object is fibrant, the composite map $S\tilde{C}TX \rightarrow STX \rightarrow X$ is in \mathbf{W} by hypothesis. \square

Corollary 2.8. *If (S, T) is a Quillen equivalence, the \mathcal{G} -cofibrant objects of \mathbf{M} coincide with the cofibrant objects of \mathbf{M} and the \mathcal{G} -fibrant objects of \mathbf{N} coincide with the fibrant objects of \mathbf{N} , then $(S^{\mathcal{G}}, T^{\mathcal{G}})$ is a Quillen equivalence.*

The next result can be seen as a generalization of the fact [5, 4.1.1(4)] that “a left Bousfield localization of a simplicial model category is a simplicial model category”. Its proof shows that *loc. cit.* is actually a formal result. We recall from [6, 4.2.6 and 4.2.18] the notions of monoidal model category and \mathcal{C} -model category. In what follows we shall neglect the second part of these definitions.

Proposition 2.9. *Suppose that \mathbf{M} is a \mathcal{V} -model category, for some cofibrantly generated monoidal model category \mathcal{V} which has a generating set of cofibrations with cofibrant domains. Let us write $X * K$ and X^K for the tensor and cotensor of $X \in \mathbf{M}$ with an object K of \mathcal{V} . Then $\mathbf{M}^{\mathcal{G}}$ is a \mathcal{V} -model category (for the same tensor and cotensor) if and only if*

(1) *for every map $A \rightarrow B$ in $\mathcal{C}^{\mathcal{G}}$ and every generating cofibration $J \rightarrow K$ of \mathcal{V} , the map*

$$A * K \coprod_{A * J} B * J \rightarrow B * K$$

is in $\mathcal{C}^{\mathcal{G}}$, and

(2) *for every map $X \rightarrow Y$ in $\mathbf{F}^{\mathcal{G}}$ between \mathcal{G} -fibrant objects and every object K belonging to the set of domains and codomains of the generating cofibrations of \mathcal{V} , the map $X^K \rightarrow Y^K$ is in $\mathbf{F}^{\mathcal{G}}$.*

If \mathbf{M} is right proper, (1) can be replaced by

(1') *for every generating cofibration $J \rightarrow K$ of \mathcal{V} and every map $X \rightarrow Y$ in $\mathbf{F} \cap \mathbf{W}^{\mathcal{G}}$ between fibrant objects, the map*

$$X^K \rightarrow X^J \prod_{Y^J} Y^K$$

is in $\mathbf{F}^{\mathcal{G}} \cap \mathbf{W}^{\mathcal{G}}$.

Proof. For the equivalence between the (first part of the) \mathcal{V} -model category axiom and (1) and (2) one uses the fact that, in a model category, a cofibration is a weak equivalence if and only if it has the left lifting property with respect to every fibration between fibrant objects [7, Lemma 7.14]. For the rest one uses [5, 13.2.1(2)]. \square

Lemma 2.10. *Suppose that \mathbf{M} is a \mathcal{V} -model category, for some cofibrantly generated monoidal model category \mathcal{V} which has a generating set of cofibrations with cofibrant domains. Let us write $X * K$ for the tensor of $X \in \mathbf{M}$ with an object K of \mathcal{V} . Assume that $\mathbf{M}^{\mathcal{G}}$ is a right Bousfield localization of \mathbf{M} . Then $\mathbf{M}^{\mathcal{G}}$ is a \mathcal{V} -model category (for the same structure) if and only if for every K belonging to the set of domains and codomains of the generating cofibrations of \mathcal{V} and every \mathcal{G} -cofibrant object A , $K * A$ is \mathcal{G} -cofibrant.*

Proof. The proof is similar to the proof of Proposition 2.9. \square

3. APPLICATION: CATEGORIES ENRICHED OVER MONOIDAL MODEL CATEGORIES

In this section we give the following application of Theorem 1.5. Let \mathcal{V} be a closed category. We denote by $\mathcal{V}\text{-Cat}$ the category whose objects are the small \mathcal{V} -categories and whose morphisms are the \mathcal{V} -functors. When \mathcal{V} is a model category satisfying certain assumptions, $\mathcal{V}\text{-Cat}$ admits the fibred model structure [8, 4.4]. Under further assumptions on \mathcal{V} , we shall exhibit in Theorem 3.1 an $\ell\ell$ -extension of the fibred model structure.

Let \mathcal{V} be a monoidal model category with cofibrant unit I . Let \mathbf{Cat} be the category of small categories. We have a functor $[-]_{\mathcal{V}} : \mathcal{V}\text{-}\mathbf{Cat} \rightarrow \mathbf{Cat}$ obtained by change of base along the symmetric monoidal composite functor

$$\mathcal{V} \longrightarrow Ho(\mathcal{V}) \xrightarrow{Ho(\mathcal{V})(I, -)} Set$$

Let \mathcal{K} be a class of maps of \mathcal{V} . We say that a \mathcal{V} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is **locally in \mathcal{K}** if for each pair x, y of objects of \mathcal{A} , the map $f_{x,y} : \mathcal{A}(x, y) \rightarrow \mathcal{B}(fx, fy)$ is in \mathcal{K} .

Definition 2. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in $\mathcal{V}\text{-}\mathbf{Cat}$.

1. The morphism f is a **weak equivalence** if f is locally a weak equivalence of \mathcal{V} and $[f]_{\mathcal{V}} : [\mathcal{A}]_{\mathcal{V}} \rightarrow [\mathcal{B}]_{\mathcal{V}}$ is essentially surjective.
2. The morphism f is a **fibration** if
 - (a) f is locally a fibration of \mathcal{V} , and
 - (b) for any $x \in Ob(\mathcal{A})$, and any isomorphism $v : y' \rightarrow [f]_{\mathcal{V}}(x)$ in $[\mathcal{B}]_{\mathcal{V}}$, there exists an isomorphism $u : x' \rightarrow x$ in $[\mathcal{A}]_{\mathcal{V}}$ such that $[f]_{\mathcal{V}}(u) = v$.
3. The morphism f is called a **cofibration** if it has the left lifting property with respect to the fibrations which are weak equivalences.

We denote by $W^{\mathcal{G}}$ the class of weak equivalences, by $C^{\mathcal{G}}$ the class of cofibrations and by $F^{\mathcal{G}}$ the class of fibrations. It follows directly from the definitions that a \mathcal{V} -functor is in $F^{\mathcal{G}} \cap W^{\mathcal{G}}$ if and only if it is surjective on objects and locally a trivial fibration of \mathcal{V} .

Let now \mathcal{V} be as in [8, 4.2]. Then the category $\mathcal{V}\text{-}\mathbf{Cat}$ admits a weak factorization system $(C^{\mathcal{G}}, F^{\mathcal{G}} \cap W^{\mathcal{G}})$ and the fibred model structure [8, 4.4]. A \mathcal{V} -category \mathcal{A} is cofibrant (fibrant) in the fibred model structure if and only if \mathcal{A} is cofibrant (fibrant) in the model structure on the category of \mathcal{V} -categories with fixed set of objects $Ob(\mathcal{A})$, and the map from the initial \mathcal{V} -category to \mathcal{A} (from \mathcal{A} to the terminal \mathcal{V} -category) is in $C^{\mathcal{G}}$ ($F^{\mathcal{G}}$) if and only if \mathcal{A} is cofibrant (fibrant) in the fibred model structure.

Theorem 3.1. *Suppose furthermore that \mathcal{V} is right proper and for every locally fibrant \mathcal{V} -category \mathcal{A} , there is a factorization*

$$\mathcal{A} \xrightarrow{s} Path\mathcal{A} \xrightarrow{p_0 \times p_1} \mathcal{A} \times \mathcal{A}$$

of the diagonal map $\mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$ such that s is in $W^{\mathcal{G}}$ and $p_0 \times p_1$ is in $F^{\mathcal{G}}$. Then the classes $W^{\mathcal{G}}, C^{\mathcal{G}}$ and $F^{\mathcal{G}}$ form a right proper model structure on $\mathcal{V}\text{-}\mathbf{Cat}$.

Proof. We take in Theorem 1.5 the category \mathbf{M} to be $\mathcal{V}\text{-}\mathbf{Cat}$ regarded as having the fibred model structure. It is not difficult to show that assumptions (1), (2), (3) and (4) hold in our case. We have already shown above that assumptions (7) and (8) also hold. Assumption (6) holds since \mathcal{V} is right proper whereas (5) is incorporated in the statement of the Theorem. \square

We said at the beginning of Section 1 that Bousfield's proofs of [2, Theorems 3.3 and 12.4] are, after a certain stage, formal. Hence the proofs of the duals of *loc. cit.* are, after a certain stage, formal. This raises the question whether one really needs our Theorem 1.5 for the proof of Theorem 3.1. We notice that the dual of [2, Lemma 2.5] does not hold in our example. Explicitly, if $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ are \mathcal{V} -functors with gf a fibration in the sense of Definition 2.2 and g a fibration in the fibred model structure, then g is not necessarily a fibration.

In order to apply Theorem 3.1 one needs to eventually construct the required factorization of the diagonal. This is not immediate. A study of closed categories for which this factorization is possible is beyond the scope of this paper. Nevertheless, examples include the categories of: small groupoids, small categories, compactly generated Hausdorff spaces, chain complexes of R -modules, simplicial R -modules (where R is a commutative ring), small 2-categories and small \mathcal{V} -categories (where \mathcal{V} is a locally presentable closed category). Work of B. van den Berg and R. Garner [1] seems to suggest a construction of the required factorization for the category of simplicial sets.

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